

Properties of information carrying waves in cosmology

E. M. O'Shea*

Mathematical Physics Department, National University of Ireland Dublin, Belfield, Dublin 4, Ireland

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Recently we studied the effects of information carrying waves propagating through isotropic cosmologies. By information carrying we mean that the waves have an arbitrary dependence on a function. We found that the waves introduce shear and anisotropic stress into the universe. We then constructed explicit examples of pure gravity wave perturbations for which the presence of this anisotropic stress is essential and the null hypersurfaces playing the role of the histories of the wavefronts in the background space-time are shear free. Motivated by this result we now prove that these two properties are true *for all information carrying waves* in isotropic cosmologies.

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I. INTRODUCTION

In a recent paper [1] the propagation of shear-free gravitational waves through isotropic universes was studied using the gauge-invariant and covariant formalism of Ellis and Bruni [2]. In this approach the waves are represented as a perturbation of the Robertson-Walker space-time. The gravitational waves studied also had the property that they could carry arbitrary information. By this we mean that the Ellis-Bruni gauge-invariant variables had an arbitrary dependence on a function.

It was found that the consistency of the partial differential equations governing the perturbations requires that all gauge-invariant perturbed quantities must vanish except for the perturbed shear of the matter world lines and the anisotropic stress of the matter distribution. A wave equation for the perturbed shear was derived. In addition the perturbed shear was shown to satisfy a propagation equation along the null rays associated with the gravitational waves [see Eqs. (2.7) and (2.8) below].

For pure gravitational radiation (i.e. the perturbed Weyl tensor is type N in the Petrov classification) explicit examples were constructed for which it was shown (using the wave equation and the propagation equation) that the presence of the anisotropic stress is essential and the hypersurfaces playing the role of the histories of the wavefronts are shear free. We now prove that these two properties are true in general and are not in any way dependent on the examples constructed.

The paper is organized as follows: The notation used and some useful equations are given in the Appendix and referred to where appropriate. In Sec. II we outline the results obtained in [1] which we use in the paper. The first of our results is also given in this section. The elements of the Newman-Penrose formalism necessary for this problem are described in Sec. III and the main result is derived in Sec. IV. The paper ends with a summary of the results of Secs. III and IV.

II. EQUATIONS GOVERNING GRAVITATIONAL WAVES

Throughout this paper we shall use the notation and sign conventions of [3]. For convenience these have been briefly outlined in the Appendix. We are concerned with a four dimensional space-time manifold with metric tensor g_{ij} in a local coordinate system x^i and a preferred congruence of world lines tangent to the unit vector field u^i (with $u^i u_i = -1$). The decompositions of the symmetric energy-momentum-stress tensor T_{ij} and the Weyl tensor with respect to this 4-velocity field, along with definitions of the kinematical quantities σ_{ij} , ω_{ij} , θ , can be found in the Appendix.

In a recent paper [1] we used the gauge-invariant and covariant approach of Ellis and Bruni [2] to construct gravitational wave perturbations of Robertson-Walker spacetimes. The background metric tensor is thus the Robertson-Walker metric, the background energy-momentum-stress tensor is Eq. (A9) specialized to a perfect fluid (i.e. with $q^i = 0 = \pi^{ij}$) with fluid 4-velocity u^i and the background Weyl tensor vanishes. The Ellis-Bruni approach involves working in a general local coordinate system with gauge-invariant small quantities rather than small perturbations of the background metric. These quantities have the property that they vanish in the background space-time. For isotropic spacetimes the Ellis-Bruni variables are σ_{ij} , \dot{u}^i , ω_{ij} , $X_i = h_i^j \mu_{,j}$, $Y_i = h_i^j p_{,j}$, $Z_i = h_i^j \theta_{,j}$, π_{ij} , q_i , E_{ij} and H_{ij} (see the Appendix for definitions of these variables). We found that it is tensor quantities that describe gravitational wave perturbations and thus we can set all Ellis-Bruni variables equal to zero except for π_{ij} , σ_{ij} , E_{ij} and H_{ij} . The equations satisfied by these variables are obtained by projections, in the direction of and orthogonal to the 4-velocity u^i , of the equations of motion and the energy conservation equation contained in $T^{ij}_{;j} = 0$, the Ricci identities and the Bianchi identities (see the Appendix). These projections give rise to a lengthy list of equations which can be found in [1].

We specialized to a particular class of gravitational waves, namely those which can be viewed as carrying arbitrary information. To incorporate this into the Ellis-Bruni setup we required that the gauge-invariant small quantities have an arbitrary dependence on a function. Specifically we assume that

*Email address: emer.oshea@ucd.ie

$$\pi_{ij} = \Pi_{ij} F(\phi), \quad \sigma_{ij} = s_{ij} F(\phi), \quad (2.1)$$

where F is an arbitrary real-valued function of its argument $\phi(x^i)$. This idea of introducing arbitrary functions into solutions of Einstein's equations describing gravitational waves goes back to work by Trautman [4] and this form for the Ellis-Bruni variables was first used by Hogan and Ellis [5]. Using the projections of the Ricci identities with $\omega_{ij}=0 = \dot{u}^i$ we can write

$$E_{ij} = \frac{1}{2} \pi_{ij} + \frac{2}{3} \sigma^2 h_{ij} - \frac{2}{3} \theta \sigma_{ij} - \sigma_{ik} \sigma^k_j - h_i^k h_j^l \dot{\sigma}_{kl} \quad (2.2)$$

and

$$H_{ij} = -h_i^k h_j^l \sigma_{(k}{}^{p;q} \eta_{l)rpq} u^r. \quad (2.3)$$

Thus these variables are derived from σ_{ij} and π_{ij} and it is not necessary to make any extra assumptions about their dependence on $F(\phi)$. We note that Π_{ij} , s_{ij} are orthogonal to u^i and trace free with respect to the background metric g_{ij} .

Substituting Eq. (2.1) into the linearized versions of the equations satisfied by these variables we find that s^{ij} and Π^{ij} are orthogonal to the gradient of ϕ and divergent free; i.e., we have

$$s^{ij} \phi_{,j} = 0, \quad \Pi^{ij} \phi_{,j} = 0 \quad (2.4)$$

and

$$s^{ij}{}_{;j} = 0, \quad \Pi^{ij}{}_{;j} = 0. \quad (2.5)$$

Also, with $\dot{\phi} := \phi_{,i} u^i \neq 0$, consistency of the equations requires that

$$g^{ij} \phi_{,i} \phi_{,j} = 0, \quad (2.6)$$

where g_{ij} is the background metric. This means that the hypersurfaces $\phi(x^i) = \text{const}$ in the background isotropic space-time playing the role of the histories of the wavefronts must be null. The wave equation for s^{ij} ,

$$\begin{aligned} s^{ij;k}{}_{;k} - \frac{2}{3} \theta \dot{s}^{ij} - \left(\frac{1}{3} \dot{\theta} + \frac{4}{9} \theta^2 \right) s^{ij} + \left(p - \frac{1}{3} \mu_0 \right) s^{ij} \\ = -\Pi^{ij} - \frac{2}{3} \theta \Pi^{ij}, \end{aligned} \quad (2.7)$$

and propagation equation for s_{ij} along the null geodesics tangent to ϕ^i ,

$$s_{ij;k} \phi^{,k} + \left(\frac{1}{2} \phi^{,k}{}_{;k} - \frac{1}{3} \theta \dot{\phi} \right) s_{ij} = -\frac{1}{2} \dot{\phi} \Pi_{ij}, \quad (2.8)$$

are consequences of the Ricci identities. We have used μ_0 here to denote the energy density of the cosmic fluid to avoid confusion with the spin coefficient μ used in the next section. The internal consistencies of these equations were checked in [1]. Substituting Eq. (2.1) into Eqs. (2.2) and (2.3) we find that the electric and magnetic parts of the Weyl tensor are now given by

$$E_{ij} = \left(\frac{1}{2} \Pi_{ij} - \dot{s}_{ij} - \frac{2}{3} \theta s_{ij} \right) F - \dot{\phi} s_{ij} F' \quad (2.9)$$

and

$$H_{ij} = -s_{(i}{}^{l;m} \eta_{j)klm} u^k F - s_{(i}{}^l \eta_{j)klm} u^k \phi^{,m} F' \quad (2.10)$$

where $F' = \partial F / \partial \phi$.

We are interested here in pure gravitational wave perturbations, i.e. perturbations which have pure type N perturbed Weyl tensor in the Petrov classification. It is easily checked using Eq. (2.4) that the F' parts of E_{ij} and H_{ij} above are automatically type N with degenerate principal null direction ϕ^i . However the F parts of E_{ij} and H_{ij} are not in general type N and so to describe pure gravitational wave perturbations we require (see [1]) that

$$\dot{s}^{ij} \phi_{,j} = 0, \quad s^{ij} \phi_{,i}{}^{;k} - s^{ik} \phi_{,i}{}^{;j} = 0. \quad (2.11)$$

Making use of the null tetrad $k_i = -\dot{\phi}^{-1} \phi_{,i}$, $l_i = u_i - \frac{1}{2} k_i$, and m_i , \bar{m}_i a complex covariant vector field and its complex conjugate chosen so that they are null ($m^i m_i = 0 = \bar{m}^i \bar{m}_i$), are orthogonal to k^i and l^i and satisfy $m^i \bar{m}_i = 1$ we find that (see [1]) the conditions (2.11) are equivalent to

$$s \phi_{,i;j} \bar{m}^i l^j = 0 \quad (2.12)$$

and

$$\bar{s} \phi_{,i;j} m^i m^j = s \phi_{,i;j} \bar{m}^i \bar{m}^j. \quad (2.13)$$

In terms of the tetrad we can write s^{ij} as (because $s^{ij} u_j = 0$, $s^i{}_i = 0$, $s^{ij} \phi_{,j} = 0$ and so $s^{ij} k_j = 0$ and $s^{ij} l_j = 0$)

$$s^{ij} = \bar{s} m^i m^j + s \bar{m}^i \bar{m}^j, \quad (2.14)$$

where $|s|^2 = \frac{1}{2} s^{ij} s_{ij}$. For future use we point out here that Π^{ij} can be expressed on the tetrad in a similar way: namely,

$$\Pi^{ij} = \bar{\Pi} m^i m^j + \Pi \bar{m}^i \bar{m}^j. \quad (2.15)$$

We emphasize that we are using Π here for the complex component of Π^{ab} to avoid confusion with the spin coefficient π appearing in the next section. Taking the divergence of the first equation of Eqs. (2.4) and substituting for s^{ij} from Eq. (2.14) yields (since $s^{ij}{}_{;j} = 0$)

$$(\bar{s} m^i m^j + s \bar{m}^i \bar{m}^j) \phi_{,j;i} = 0. \quad (2.16)$$

Comparing this with Eq. (2.13) it is clear that for consistency (with $s \neq 0$) we must have

$$\phi_{,i;j} m^i m^j = 0. \quad (2.17)$$

Thus the hypersurfaces $\phi(x^i) = \text{const}$ which play the role of the histories of the wavefronts must be shear free. With $s \neq 0$, Eq. (2.12) becomes

$$\phi_{,i;j} \bar{m}^i l^j = 0. \quad (2.18)$$

This condition corresponds to the vanishing of the spin coefficients τ, ν which we shall make use of in the next section.

III. NEWMAN-PENROSE FORMALISM

For the remainder of the paper we seek to demonstrate that the presence of anisotropic stress in the perturbed space-time is essential for the existence of pure gravitational waves. We find the most effective way to do this is to make use of the Newman-Penrose formalism. A comprehensive description of this formalism can be found in [6]. Specifically we choose the tetrad described after Eq. (2.11) and we label the vectors as follows:

$$\begin{aligned} D &= k^i \partial_i = e_1^i \partial_i, & \Delta &= l^i \partial_i = e_2^i \partial_i, \\ \delta &= m^i \partial_i = e_3^i \partial_i, & \delta^* &= \bar{m}^i \partial_i = e_4^i \partial_i. \end{aligned} \quad (3.1)$$

The spin coefficients (Ricci rotation coefficients) are denoted by $\gamma_{abc} = -\gamma_{bac}$. It follows from Eq. (2.17) that k^i is shear free (i.e. $k_{i;j} m^i m^j = 0$) and a simple calculation using the fact that we can write $\dot{\phi}_{,b} = (\phi_{,b}) + \frac{1}{3} \theta h_b^c \phi_{,c}$ shows that k^i is also geodesic with $k^i_{;j} k^j = \frac{1}{3} \theta k^i$. As a consequence of these properties the spin coefficients κ, σ, λ and π vanish. We again point out that π here is not to be confused with the complex component Π of Π^{ij} . As previously mentioned after Eq. (2.18) the spin coefficients ν and τ also vanish. Therefore, for the problem at hand the only non-zero background spin coefficients are $\rho, \mu, \gamma, \epsilon, \alpha, \beta$ with

$$\rho = -\frac{1}{2} \dot{\phi}^{-1} \phi^a_{;a}, \quad (3.2)$$

$$\mu = \frac{1}{2} \rho - \frac{1}{3} \theta, \quad (3.3)$$

$$\epsilon = \frac{1}{2} (\gamma_{211} + \gamma_{341}), \quad (3.4)$$

$$\gamma = \frac{1}{2} (\gamma_{212} + \gamma_{342}), \quad (3.5)$$

$$\alpha = \frac{1}{2} (\gamma_{214} + \gamma_{344}), \quad (3.6)$$

$$\beta = \frac{1}{2} (\gamma_{213} + \gamma_{343}). \quad (3.7)$$

Using $\gamma_{abc} = -\gamma_{bac}$ and noting that the 4-acceleration vanishes in the background we find that

$$\gamma + \gamma^* = \frac{1}{6} \theta, \quad \epsilon + \epsilon^* = -\frac{1}{3} \theta, \quad (3.8)$$

where, following Chandrasekhar, the star denotes complex conjugation. In addition ϕ is a real-valued function and thus ρ and μ are real and we have

$$\rho - \rho^* = 0, \quad \mu - \mu^* = 0. \quad (3.9)$$

We will use these facts in the next section when we examine the wave equation (2.7) and the propagation equation (2.8) in this formalism.

In the tetrad formalism the Ricci identities (A7) read

$$\begin{aligned} R_{abcd} &= -\gamma_{abc,d} + \gamma_{abd,c} + \gamma_{baf}(\gamma_c^f{}_d - \gamma_d^f{}_c) + \gamma_{fac}\gamma_b^f{}_d \\ &\quad - \gamma_{fad}\gamma_b^f{}_c, \end{aligned} \quad (3.10)$$

where R_{abcd} are the tetrad components of the Riemann tensor. An alternative expression for these components is given via the definition of the tetrad components of the Weyl tensor: namely,

$$\begin{aligned} R_{abcd} &= C_{abcd} + \frac{1}{2} (\eta_{ac} R_{bd} - \eta_{bc} R_{ad} - \eta_{ad} R_{bc} + \eta_{bd} R_{ac}) \\ &\quad - \frac{1}{6} (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}) R. \end{aligned} \quad (3.11)$$

Here $R_{ab} = R_{ij} e_a^i e_b^j$ are the tetrad components of the Ricci tensor, $R = \mu_0 - 3p$ is the Ricci scalar and $\eta_{ab} := e_a^i e_{ib} = 0$ except for $\eta_{12} = \eta_{21} = -\eta_{34} = -\eta_{43} = -1$. In general we label tetrad components of tensors using a, b, c, \dots and coordinate components using i, j, k, \dots . Making use of Eqs. (A8) and (A9) we find that the tetrad components of the background perfect fluid can be written in the form

$$R_{ab} = (\mu_0 + p) u_a u_b + \frac{1}{2} (\mu_0 - p) \eta_{ab}, \quad (3.12)$$

where $u_a = u_i e_a^i$ with $u_i = l_i + \frac{1}{2} k_i$ and thus $u_a = (-1, -\frac{1}{2}, 0, 0)$. Here u_a are the tetrad components of the 4-velocity of a fluid particle, μ_0 is the proper density and p is the isotropic pressure. We shall always assume that $\mu_0 + p \neq 0$ so that R_{ab} has a unique time-like eigenvector. Evaluating the various components of the Riemann tensor using both Eqs. (3.10) and (3.11) (we use the commutation relations involving the operators $D, \Delta, \delta, \delta^*$ to do this) and then equating the two different expressions obtained yields the following 12 equations for the derivatives of the *background* spin coefficients:

$$D\rho = -\rho^2 - (\epsilon + \epsilon^*) - \frac{1}{2} (\mu_0 + p), \quad (3.13)$$

$$D\alpha - \delta^* \epsilon = -\alpha(\rho + \epsilon^* - 2\epsilon) + \beta^* \epsilon, \quad (3.14)$$

$$D\beta - \delta \epsilon = \beta(\epsilon^* - \rho^*) + \epsilon \alpha^*, \quad (3.15)$$

$$D\gamma - \Delta \epsilon = 2\epsilon\gamma + \epsilon\gamma^* + \epsilon^*\gamma - \frac{1}{12} \mu_0 - \frac{1}{4} p, \quad (3.16)$$

$$D\mu = -\mu\rho^* + \mu(\epsilon + \epsilon^*) + \frac{1}{4} p - \frac{1}{12} \mu_0, \quad (3.17)$$

$$\delta\rho = -\rho(\alpha^* + \beta), \quad (3.18)$$

$$\delta\alpha - \delta^*\beta = -\mu\rho - \alpha\alpha^* - \beta\beta^* + 2\alpha\beta - \frac{1}{6}\mu_0, \quad (3.19)$$

$$\delta^*\mu = \mu(\alpha + \beta^*), \quad (3.20)$$

$$\Delta\mu = \mu^2 + \mu(\gamma + \gamma^*) + \frac{1}{8}(\mu_0 + p), \quad (3.21)$$

$$\delta\gamma - \Delta\beta = 2\beta\gamma + \alpha^*\gamma - \mu\beta - \beta\gamma^*, \quad (3.22)$$

$$\Delta\rho = \rho\mu^* - \rho(\gamma + \gamma^*) - \frac{1}{4}p + \frac{1}{12}\mu_0, \quad (3.23)$$

$$\Delta\alpha - \delta^*\gamma = \mu^*\alpha - \beta^*\gamma - \alpha\gamma^*. \quad (3.24)$$

As a result of the fact that so many of the background spin coefficients are zero, the remaining Ricci identities give no further information. We note that we have made use of the vanishing of the background Weyl tensor in this calculation. Also for clarity we refrain from substituting for the spin coefficients from Eqs. (3.2)–(3.7) until the very end.

Under the (class III) rotation

$$k^j \rightarrow k^j, \quad l^j \rightarrow l^j, \quad m^j \rightarrow e^{i\varphi} m^j, \quad \bar{m}^j \rightarrow e^{-i\varphi} \bar{m}^j, \quad (3.25)$$

where φ is a real function, the non-zero spin coefficients transform as

$$\rho \rightarrow \rho, \quad (3.26)$$

$$\mu \rightarrow \mu, \quad (3.27)$$

$$\gamma \rightarrow \gamma + \frac{1}{2}i\Delta\varphi, \quad (3.28)$$

$$\epsilon \rightarrow \epsilon + \frac{1}{2}iD\varphi, \quad (3.29)$$

$$\alpha \rightarrow e^{-i\varphi}\alpha + \frac{1}{2}ie^{-i\varphi}\delta^*\varphi, \quad (3.30)$$

$$\beta \rightarrow e^{i\varphi}\beta + \frac{1}{2}ie^{i\varphi}\delta\varphi. \quad (3.31)$$

From these transformations we see that we can in principle rotate away $\epsilon - \epsilon^*$, $\gamma - \gamma^*$ and $\alpha - \beta^*$ (that is, we can assume ϵ, γ are real and $\alpha = \beta^*$) provided the function φ satisfies

$$D\varphi = i(\epsilon - \epsilon^*), \quad \Delta\varphi = i(\gamma - \gamma^*), \quad \delta\varphi = i(\alpha - \beta^*). \quad (3.32)$$

However, before we impose these conditions we need to check that they are consistent with each other and with all other equations. Suppose the first of these two conditions hold. Using Eq. (3.16) we find

$$D\Delta\varphi - \Delta D\varphi = 2i(\epsilon\gamma - \epsilon^*\gamma^*) \quad (3.33)$$

and it is straightforward to check that this agrees with one of the standard commutation relations. Thus we can set $\epsilon - \epsilon^* = 0$ and $\gamma - \gamma^* = 0$ without imposing any extra conditions. If we assume that the third equation of Eqs. (3.32) holds, then Eq. (3.19) and the commutation relation involving $[\delta, \delta^*]$ are consistent only if we have

$$\mu^*\rho^* + \mu\rho = -\frac{1}{3}\mu_0. \quad (3.34)$$

Since μ, ρ are real, this simplifies to

$$2\mu\rho = -\frac{1}{3}\mu_0. \quad (3.35)$$

Replacing μ here by Eq. (3.3) yields the quadratic equation

$$3\rho^2 - 2\theta\rho + \mu_0 = 0. \quad (3.36)$$

In order to have ρ real we require

$$\theta^2 \geq 3\mu_0. \quad (3.37)$$

Here θ is the expansion of the universe with $\theta = 3\dot{R}/R$ where $R(t)$ is the scale factor of the general Robertson-Walker line element and the overdot indicates differentiation with respect to t . Using the background Einstein equations $G_{ij} = (\mu_0 + p)u_i u_j + p g_{ij}$, where G_{ij} is the Einstein tensor, we obtain the background Friedmann equation

$$\theta^2 = 3\mu_0 - \frac{9k}{R(t)^2}, \quad (3.38)$$

where $k = 0, \pm 1$ is the Gaussian curvature of the homogeneous space-like hypersurfaces $t = \text{const}$. Comparing this with Eq. (3.37) we see that if we impose Eq. (3.34) we cannot continue to study universes with $k = +1$. However, in [1] we constructed examples of information carrying waves propagating through such universes and so we shall not insist on having $\alpha - \beta^* = 0$. Thus from this point we assume only that we can set $\gamma - \gamma^* = 0$ and $\epsilon - \epsilon^* = 0$ without loss of generality.

We complete this section by listing (in terms of the spin coefficients) some derivatives of the tetrad vectors which we use extensively in the next section:

$$m_{i;j}m^j = (\alpha^* - \beta)m_i, \quad (3.39)$$

$$\bar{m}_{i;j}m^j = (\alpha - \beta^*)m_j + (\beta - \alpha^*)\bar{m}_j, \quad (3.40)$$

$$m_{i;j}\bar{m}^j = -\mu^*k_i + \rho l_i + (\beta^* - \alpha)m_i, \quad (3.41)$$

$$k_{i;j}l^j = -(\gamma + \gamma^*)k_i, \quad (3.42)$$

$$m_{i;j}k^j = 0, \quad (3.43)$$

$$m_{i;j}l^j = 0. \quad (3.44)$$

IV. KEY RESULT

We now have all the equations necessary to derive our result from the wave equation (2.7) for s^{ij} using the propagation equation (2.8) and the divergence-free condition (2.5). To begin with we examine the divergence-free condition. Writing this condition in tetrad formalism using Eq. (2.14) and then contracting with m_i yields

$$\delta^* s = -2(\alpha - \beta^*)s. \quad (4.1)$$

Thus we now have an expression for the derivative of s in the direction of \bar{m}^i . In terms of the spin coefficients we can write

$$\theta = \frac{3}{2}\rho - 3\mu \quad (4.2)$$

and

$$\phi_{,i}{}^i = -2\phi\rho. \quad (4.3)$$

Multiplying the propagation equation (2.8) by $m^i m^j$ and using these expressions we arrive at (since $\phi_{,i} = -\phi k_i$) the following equation for the derivative of s in the direction of k^i :

$$Ds = \left(\mu - \frac{3}{2}\rho \right) s + \frac{1}{2}\Pi. \quad (4.4)$$

Similarly we multiply the wave equation (2.7) by $m^i m^j$ and this yields

$$\begin{aligned} s^{ij;k}{}_{;k} m_i m_j - \frac{2}{3} \theta s^{ij} m_i m_j - \left(\frac{1}{3} \dot{\theta} + \frac{4}{9} \theta^2 \right) s + \left(\rho - \frac{1}{3} \mu_0 \right) s \\ = -\dot{\Pi}^{ij} m_i m_j - \frac{2}{3} \theta \Pi. \end{aligned} \quad (4.5)$$

Writing $s^{ij;k}{}_{;k}$ as $g^{kl} s^{ij}{}_{;kl}$ and using Eq. (2.14) we can [on substitution from Eqs. (3.39)–(3.44)] write the first term of this wave equation as

$$\begin{aligned} s^{ij;k}{}_{;k} m_i m_j = 2[\delta(\delta^* s) - \Delta(Ds)] - 2(\gamma + \gamma^* - \mu)Ds \\ + 2s\delta(\alpha - \beta^*) + 2(\beta - \alpha^*)\delta^* s + 4(\alpha - \beta^*)\delta s + 2s\delta^*(\beta - \alpha^*) - 2\rho^* \Delta s + 8(\beta - \alpha^*)(\alpha - \beta^*)s - 2(\mu^* \rho^* + \mu\rho)s. \end{aligned} \quad (4.6)$$

Replacing Ds here by Eq. (4.4), $\delta^* s$ by Eq. (4.1) and using Eq. (3.19) to write $\delta^* \alpha - \delta\beta$ (and its complex conjugate) in terms of undifferentiated spin coefficients this equation simplifies to

$$\begin{aligned} s^{ij;k}{}_{;k} m_i m_j = \left[-2\Delta\mu + 3\Delta\rho - 2\left(\frac{1}{6}\theta - \mu \right) \left(\mu - \frac{3}{2}\rho \right) \right] s \\ + \frac{2}{3}\theta\Delta s + \frac{2}{3}\mu_0 s - \left(\frac{1}{6}\theta - \mu \right) \Pi - \Delta\Pi. \end{aligned} \quad (4.7)$$

We have also used Eqs. (3.8), (3.9) and (4.2) to write the equation in this form.

We now turn our attention to the second term of the wave equation (4.5). As a consequence of Eq. (2.14) we have

$$\dot{s}^{ij} m_i m_j = \dot{s} + 2s\dot{\bar{m}}^i m_i, \quad (4.8)$$

where the overdot indicates covariant differentiation in the direction of the 4-velocity u^i . Writing out this derivative explicitly (i.e. writing $\dot{s} = s_{;i} u^i$) and noting that $u^i = l^i + \frac{1}{2}k^i$ the above equation becomes (on account of $\gamma - \gamma^* = \epsilon - \epsilon^* = 0$)

$$\dot{s}^{ij} m_i m_j = \Delta s + \frac{1}{2}Ds. \quad (4.9)$$

The anisotropic stress Π^{ij} satisfies the same equations as s^{ij} [cf. Eqs. (2.4) and (2.5)] and thus we also have

$$\dot{\Pi}^{ij} m_i m_j = \Delta\Pi + \frac{1}{2}D\Pi. \quad (4.10)$$

Now reconstructing the wave equation (4.5) using Eqs. (4.7), (4.9) and (4.10) yields

$$\begin{aligned} \left[3\Delta\rho - 2\Delta\mu - 2\left(\frac{1}{6}\theta - \mu \right) \left(\mu - \frac{3}{2}\rho \right) - \frac{1}{3}\theta \left(\mu - \frac{3}{2}\rho \right) \right] s \\ + \left(p + \frac{1}{3}\mu_0 \right) s - \left(\frac{1}{3}\dot{\theta} + \frac{4}{9}\theta^2 \right) s \\ = -\frac{1}{2}D\Pi - \frac{1}{3}\theta\Pi - \mu_0\Pi. \end{aligned} \quad (4.11)$$

The background Raychaudhuri equation is

$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{1}{2}(\mu_0 + 3p). \quad (4.12)$$

Making use of this, the expressions for $\Delta\mu$ and $\Delta\rho$ given by Eqs. (3.21) and (3.23) respectively and Eq. (4.2) we find that the wave equation (4.11) reduces to the remarkably simple form

$$(\mu_0 + p)s = -D\Pi - \rho\Pi. \quad (4.13)$$

The necessity for $\Pi \neq 0$ follows from this equation because it is immediately clear that if the perturbed anisotropic stress vanishes ($\Leftrightarrow \Pi = 0$), then the perturbed shear is also zero ($\Leftrightarrow s = 0$) if $\mu_0 + p \neq 0$ and we have no perturbations (i.e. no gravitational waves). We note that the right hand side of Eq. (4.13) vanishes if Π satisfies the differential equation $D\Pi + \rho\Pi = 0$. In this case we must have $s = 0$ (since $\mu_0 + p \neq 0$) and thus from the propagation equation (4.4) we must have $\Pi = 0$.

V. SUMMARY

We have investigated the general properties of some gravitational wave perturbations of isotropic cosmologies. Specifically

we have considered gravitational waves carrying arbitrary information. By this we mean that the perturbations describing the waves have an arbitrary dependence on a real-valued function $\phi(x^i)$. Our main result hinges on two important properties: in order to have pure gravity waves the hypersurfaces $\phi(x^i)$ must be *null* and *shear free*. Now our main result (proved in Sec. IV) is that under the physically reasonable assumption that $\mu_0 + p \neq 0$ (where μ_0 and p are the proper density and pressure of the cosmic fluid) the perturbations describing the gravitational waves *must* be accompanied by the presence of anisotropic stress in cosmic fluid.

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APPENDIX: NOTATION AND BASIC EQUATIONS

Some notation and tensor quantities associated with the vector field u^i in Sec. II are required. Covariant differentiation is indicated with a semicolon and covariant differentiation in the direction of u^i is denoted by an overdot. As usual square brackets denote skew symmetrization and round brackets denote symmetrization. Thus the 4-acceleration of the time-like congruence is

$$\dot{u}^i := u^i{}_{;j} u^j. \quad (\text{A1})$$

With respect to u^i and using the projection tensor

$$h_{ij} := g_{ij} + u_i u_j, \quad (\text{A2})$$

the covariant derivative of the 4-velocity $u_{i;j}$ can be decomposed into

$$u_{i;j} = \omega_{ij} + \sigma_{ij} + \frac{1}{3} \theta h_{ij} - \dot{u}_i u_j, \quad (\text{A3})$$

with

$$\omega_{ij} := u_{[i;j]} + \dot{u}_{[i} u_{j]}, \quad (\text{A4})$$

the vorticity tensor of the congruence,

$$\sigma_{ij} := u_{(i;j)} + \dot{u}_{(i} u_{j)} - \frac{1}{3} \theta h_{ij}, \quad (\text{A5})$$

the shear tensor of the congruence and

$$\theta := u^i{}_{;i}, \quad (\text{A6})$$

the expansion (or contraction) of the universe.

The Riemann curvature tensor R_{ijkl} is defined by the Ricci identities

$$u_{i;lk} - u_{i;kl} = R_{ijkl} u^j, \quad (\text{A7})$$

and Einstein's field equations take the form

$$R_{ij} - \frac{1}{2} g_{ij} R = T_{ij}, \quad (\text{A8})$$

where $R_{ij} := R^k{}_{ikj}$ are the components of the Ricci tensor, $R := R^i{}_i$ is the Ricci scalar and T_{ij} is the symmetric energy-momentum-stress tensor. We note that the coupling constant has been absorbed into T_{ij} . With respect to the 4-velocity field u^i the energy-momentum-stress tensor can be decomposed as

$$T^{ij} = \mu_0 u^i u^j + p h^{ij} + q^i u^j + q^j u^i + \pi^{ij}, \quad (\text{A9})$$

with

$$q^i u_i = 0, \quad \pi^{ij} u_j = 0, \quad \pi^i{}_i = 0, \quad (\text{A10})$$

and $\pi^{ij} = \pi^{ji}$. Then μ_0 is interpreted as the energy density measured by an observer with 4-velocity u^i , q^i is the energy flow (such as heat flow) measured by this observer, p is the isotropic pressure and $\pi^i{}_j$ is the trace-free anisotropic stress (due for example to viscosity).

With respect to u^i the Weyl tensor may be decomposed into its ‘‘electric’’ and ‘‘magnetic’’ components given respectively by

$$E_{ij} = C_{ikjl} u^k u^l, \quad H_{ij} = *C_{ikjl} u^k u^l, \quad (\text{A11})$$

where $*C_{ikjl} = \frac{1}{2} \eta_{ik}{}^{mn} C_{mnjl}$ is the dual of the Weyl tensor (the left and right duals being equal), $\eta_{ijkl} = \sqrt{-g} \epsilon_{ijkl}$ with $g = \det(g_{ij})$ and ϵ_{ijkl} is the Levi-Civita permutation symbol. The explicit expression for the Weyl tensor in terms of E_{ij} and H_{ij} is not needed here but can be found in [3].

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